A DETERMINANT OF GENERALIZED FIBONACCI NUMBERS

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Abstract. We evaluate a determinant of generalized Fibonacci numbers, thus providing a common generalization of several determinant evaluation results that have previously appeared in the literature, all of them extending Cassini's identity for Fibonacci numbers.

1. Introduction

The well-known *Fibonacci sequence* is given by $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$. Numerous properties of this sequence are known. We refer the reader to the monograph [9] for a wealth of information on this sequence. One of these properties is the so called Cassini identity, given by

$$f_n f_{n+2} - f_{n+1}^2 = (-1)^n,$$

which can be written in matrix form as

$$\det \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} = (-1)^n. \tag{1.1}$$

Miles [6] introduced *k-generalized Fibonacci numbers* $f_n^{(k)}$ by

$$f_n^{(k)} = \sum_{i=0}^{k} f_{n-i}^{(k)},$$

with $f_n^{(k)} = 0$ for every $0 \le n \le k-2$, $f_{k-1}^{(k)} = 1$, and he gave the following generalization of (1.1):

$$\det \begin{pmatrix} f_n^{(k)} & f_{n+1}^{(k)} & \cdots & f_{n+k-1}^{(k)} \\ f_{n+1}^{(k)} & f_{n+2}^{(k)} & \cdots & f_{n+k}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+k-1}^{(k)} & f_{n+k}^{(k)} & \cdots & f_{n+2k-2}^{(k)} \end{pmatrix} = (-1)^{\frac{(2n+k)(k-1)}{2}}. \tag{1.2}$$

More recently, Stakhov [8] has generalized Cassini's identity for sequences of the form $f_n = f_{n-1} + f_{n-p-1}$.

Hoggat and Lind [4] consider the so called "dying rabbit problem", previously introduced in [1] and studied in [2] or [3], which modifies the original Fibonacci setting by letting rabbits die. In previous work by one of the authors [7], the sequence arising in

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this setting was studied in detail. For instance, the recurrence relation for this sequence depends on two parameters $k,\ell \geq 2$ and is given by

$$C_n^{(k,\ell)} = C_{n-\ell}^{(k,\ell)} + C_{n-\ell-1}^{(k,\ell)} + \cdots + C_{n-k-\ell+1}^{(k,\ell)},$$

where $C_0^{(k,\ell)},\dots,C_{k+\ell-2}^{(k,\ell)}$ are initial values which will be specified below. It was also proved that, if $r_1,\dots,r_{k+\ell-1}$ are the distinct roots of $g_{k,\ell}(x) = x^{k+\ell-1} - \frac{x^k-1}{x-1}$, then the general term of the sequence is given by $C_n^{(k,\ell)} = \sum_{i=1}^{k+\ell-1} a_i r_i$, with

$$a_i = \frac{(-1)^{k+\ell+i-1}}{\prod_{j>i}(r_j - r_i)\prod_{j<i}(r_i - r_j)} \times \left(\sum_{l=0}^{k-2} C_l^{(k,\ell)} \frac{r_i^{l+1}-1}{r_i^{l+1}(r_i-1)} + \sum_{l=k-1}^{k+\ell-3} C_l^{(k,\ell)} \frac{r_i^k - 1}{r_i^{l+1}(r_i-1)} + C_{k+\ell-2}^{(k,\ell)}\right). \tag{1.3}$$

Given the previous sequence, for every $j \geq 0$ we can define a matrix $A_{j,k,\ell}$ by

$$A_{j,k,\ell} = \begin{pmatrix} C_j^{(k,\ell)} & C_{j+\ell}^{(k,\ell)} & C_{j+\ell+1}^{(k,\ell)} & \cdots & C_{j+k+2\ell-3}^{(k,\ell)} \\ C_{j+1}^{(k,\ell)} & C_{j+\ell+1}^{(k,\ell)} & C_{j+\ell+2}^{(k,\ell)} & \cdots & C_{j+k+2\ell-2}^{(k,\ell)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{j+k+\ell-2}^{(k,\ell)} & C_{j+k+2\ell-2}^{(k,\ell)} & C_{j+k+2\ell-1}^{(k,\ell)} & \cdots & C_{j+2k+3\ell-5}^{(k,\ell)}. \end{pmatrix}$$

The main goal of this paper will be to find an explicit expression for $\det(A_{j,k,\ell})$, thus extending (1.1) and (1.2).

2. Extending Cassini's identity

Before we proceed, we have to fix our initial conditions. In the original setting [7], when we start with a pair of rabbits that become mature ℓ months after their birth and die k months after their matureness, the $k+\ell-1$ initial conditions are given by $C_0^{(k,\ell)} = \cdots = C_{\ell-1}^{(k,\ell)} = 1$ and $C_n^{(k,\ell)} = C_{n-1}^{(k,\ell)} + C_{n-\ell}^{(k,\ell)}$ for every $\ell \leq n \leq k+\ell-2$. Instead, in what follows we will consider the following initial conditions:

$$\widetilde{C}_0^{(k,\ell)} = 1,$$
$$\widetilde{C}_1^{(k,\ell)} = \cdots = \widetilde{C}_{k-1}^{(k,\ell)} = 0,$$
$$\widetilde{C}_k^{(k,\ell)} = \cdots = \widetilde{C}_{k+\ell-2}^{(k,\ell)} = 1.$$

Note that this change in the initial conditions results only in a shift of indices. Namely, if $C_n^{(k,\ell)}$ denotes the original sequence and $\widetilde{C}_n^{(k,\ell)}$ denotes the sequence given by the same recurrence relation and these new initial conditions, then for every $n \geq 0$ we have

$$C_n^{(k,\ell)} = \widetilde{C}_{n+k+1}^{(k,\ell)}.$$

Thus, if $\widetilde{A}_{j,k,\ell}$ is the corresponding matrix (defined in the obvious way), we have $A_{j,k,\ell} = \widetilde{A}_{j+k+1,k,\ell}$. Hence, we can focus on finding a formula for $\det(\widetilde{A}_{j,k,\ell})$.

First of all, observe that $\det(\widetilde{A}_{j,k,\ell}) = (-1)^{k+\ell-2}\det(\widetilde{A}_{j-1,k,\ell})$ because $\widetilde{A}_{j,k,\ell}$ is obtained from $\widetilde{A}_{j-1,k,\ell}$ by replacing the first row by the sum of the first k rows of the matrix, and then permuting the rows so that the first row becomes the last one. If we apply this idea repeatedly, we obtain that $\det(\widetilde{A}_{j,k,\ell}) = (-1)^{j(k+\ell-2)}\det(\widetilde{A}_{0,k,\ell})$. Hence, it is sufficient to compute this latter determinant.

We shall focus now on computing this determinant, which explicitly is

$$\det(\widetilde{A}_{0,k,\ell}) = \det\begin{pmatrix} \widetilde{C}_0^{(k,\ell)} & \widetilde{C}_\ell^{(k,\ell)} & \widetilde{C}_{\ell+1}^{(k,\ell)} & \cdots & \widetilde{C}_{k+2\ell-3}^{(k,\ell)} \\ \widetilde{C}_1^{(k,\ell)} & \widetilde{C}_{\ell+1}^{(k,\ell)} & \widetilde{C}_{\ell+2}^{(k,\ell)} & \cdots & \widetilde{C}_{k+2\ell-2}^{(k,\ell)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{C}_{k+\ell-2}^{(k,\ell)} & \widetilde{C}_{k+2\ell-2}^{(k,\ell)} & \widetilde{C}_{k+2\ell-1}^{(k,\ell)} & \cdots & \widetilde{C}_{2k+3\ell-5}^{(k,\ell)} \end{pmatrix}.$$

To do so, recall that we have $\widetilde{C}_n^{(k,\ell)} = \sum_{s=1}^{k+\ell-1} a_s r_s^n$, where the a_i's are given by (1.3). We substitute this in the above determinant and use multilinearity in the columns to expand it into the sum

$$\sum_{1\le s_1,\dots,s_{k+\ell-1}\le k+\ell-1}\left(\prod_{j=1}^{k+\ell-1} a_{s_j}\right)\det_{1\le i\le k+\ell-1}\left(r_{s_1}^{i-1}\ r_{s_2}^{i+\ell-1}\ r_{s_3}^{i+\ell}\ \cdots\ r_{s_{k+\ell-1}}^{i+k+2\ell-4}\right).$$

Now, if in this sum two of the s_j's should equal each other, then the corresponding two columns in the determinant would be dependent so that the determinant would vanish. We can therefore restrict the sum to permutations of $\{1,2,\dots,k+\ell-1\}$. With $S_{k+\ell-1}$ denoting the set of these permutations, this leads to

$$\begin{aligned}
\det(\widetilde{A}_{0,k,\ell}) &= \sum_{\sigma\in S_{k+\ell-1}}\left(\prod_{j=1}^{k+\ell-1} a_{\sigma(j)}\right)\det_{1\le i\le k+\ell-1}\left(r_{\sigma(1)}^{i-1}\ r_{\sigma(2)}^{i+\ell-1}\ r_{\sigma(3)}^{i+\ell}\ \cdots\ r_{\sigma(k+\ell-1)}^{i+k+2\ell-4}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\sum_{\sigma\in S_{k+\ell-1}}\left(\prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{\ell+j-2}\right)\det_{1\le i,j\le k+\ell-1}\left(r_{\sigma(j)}^{i-1}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\sum_{\sigma\in S_{k+\ell-1}}(\operatorname{sgn}\sigma)\left(\prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{\ell+j-2}\right)\det_{1\le i,j\le k+\ell-1}\left(r_j^{i-1}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)\right)\sum_{\sigma\in S_{k+\ell-1}}(\operatorname{sgn}\sigma)\left(\prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{\ell+j-2}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)\right)\det_{1\le i\le k+\ell-1}\left(1\ r_i^{\ell}\ r_i^{\ell+1}\ \cdots\ r_i^{k+2\ell-3}\right)\\
&= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)\right)\left(\prod_{i=1}^{k+\ell-1} r_i\right)^{k+2\ell-3}\\
&\qquad\times\det_{1\le i\le k+\ell-1}\left(r_i^{-k-2\ell+3}\ r_i^{-k-\ell+3}\ r_i^{-k-\ell+4}\ \cdots\ 1\right)
\end{aligned}$$

$$= \left(\prod_{j=1}^{k+\ell-1} a_j\right)\left(\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)(r_i^{-1}-r_j^{-1})\right)\left(\prod_{i=1}^{k+\ell-1} r_i\right)^{k+2\ell-3}$$
$$\times h_{\ell-1}(r_1^{-1},\dots,r_{k+\ell-1}^{-1}). \tag{2.1}$$

In the last line we have used the following notations and facts: first of all, $h_m(x_1,\dots,x_N)$ denotes the m-th complete homogeneous symmetric function in N variables $x_1,\dots,x_N$, explicitly given by

$$h_m(x_1,\dots,x_N)=\sum_{1\le i_1\le\dots\le i_m\le N} x_{i_1}\cdots x_{i_m}.$$

Furthermore, the Schur function indexed by a partition $\lambda=(\lambda_1,\dots,\lambda_N)$ in the variables $x_1,\dots,x_N$ is defined by

$$s_\lambda(x_1,\dots,x_N)=\frac{\det_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j}\right)}{\det_{1\le i,j\le N}\left(x_i^{N-j}\right)}=\frac{\det_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j}\right)}{\prod_{1\le i<j\le N}(x_i-x_j)}.$$

It is well-known (cf. [5, p. 41, Eq. (3.4)]) that for $\lambda=(m,0,\dots,0)$ the Schur function $s_\lambda(x_1,\dots,x_N)$ reduces to $h_m(x_1,\dots,x_N)$. These facts together explain the last line in the above computation.

To proceed further, let us first observe that, by reading off the constant coefficient of $g_{k,\ell}(x)$, we obtain

$$\prod_{i=1}^{k+\ell-1} r_i=(-1)^{k+\ell}.$$

Furthermore, we have

$$g_{k,\ell}(x)=x^{k+\ell-1}-\frac{x^k-1}{x-1}=\prod_{i=1}^{k+\ell-1}(x-r_i)=(-1)^{k+\ell-1}\prod_{i=1}^{k+\ell-1} r_i(1-r_i^{-1}x)$$
$$=-\prod_{i=1}^{k+\ell-1}(1-r_i^{-1}x).$$

Hence, we obtain

$$\sum_{m=0}^{\infty} h_m(r_1^{-1},\dots,r_{k+\ell-1}^{-1})x^m=\frac{1}{\prod_{i=1}^{k+\ell-1}(1-r_i^{-1}x)}$$
$$=\frac{1}{\frac{x^k-1}{x-1}-x^{k+\ell-1}}$$
$$=\frac{1-x}{1-x^k-x^{k+\ell-1}+x^{k+\ell}}$$
$$=1-x+x^k-x^{k+1}+\dots+O(x^{k+\ell-1}).$$

In order to evaluate $h_{\ell-1}(r_1^{-1},\dots,r_{k+\ell-1}^{-1})$, we just have to extract the coefficient of $x^{\ell-1}$ in the expansion on the right-hand side. This is easy: if $\ell-1$ equals a multiple of k

then we obtain 1, if $\ell-2$ equals a multiple of k then we obtain -1, and in all other cases we obtain 0.

We continue evaluating the other factors in (2.1). We have

$$\begin{aligned}\prod_{1\le i<j\le k+\ell-1}(r_j-r_i)(r_i^{-1}-r_j^{-1}) &= \prod_{1\le i<j\le k+\ell-1}\frac{(r_j-r_i)^2}{r_ir_j}\\ &= \frac{\prod\limits_{1\le i<j\le k+\ell-1}(r_j-r_i)^2}{\left(\prod\limits_{i=1}^{k+\ell-1} r_i\right)^{k+\ell-2}}\\ &= \frac{\prod\limits_{1\le i<j\le k+\ell-1}(r_j-r_i)^2}{(-1)^{k+\ell}}.\end{aligned}$$

Furthermore, we must compute $\prod\limits_{j=1}^{k+\ell-1} a_j$. To begin with, recall the formula (1.3) and the fact that $\widetilde{C}_0^{(k,\ell)}=\widetilde{C}_k^{(k,\ell)}=\cdots=\widetilde{C}_{k+\ell-2}^{(k,\ell)}=1$ and $\widetilde{C}_1^{(k,\ell)}=\cdots=\widetilde{C}_{k-1}^{(k,\ell)}=0$. With this in mind, we get

$$\begin{aligned}\prod_{j=1}^{k+\ell-1} a_j &= \frac{\prod\limits_{j=1}^{k+\ell-1}(-1)^{k+\ell+j-1}}{\prod\limits_{1\le i<j\le k+\ell-1}(r_j-r_i)^2}\prod_{j=1}^{k+\ell-1}\left(\frac{r_j-1}{r_j(r_j-1)}+\sum_{i=1}^{\ell-2}\frac{r_j^k-1}{r_j^{k+i}(r_j-1)}+1\right)\\ &= \frac{(-1)^{\frac{(3k+3\ell-2)(k+\ell-1)}{2}}}{\prod\limits_{1\le i<j\le k+\ell-1}(r_j-r_i)^2}\prod_{j=1}^{k+\ell-1}\left(\frac{1}{r_j}+\sum_{i=1}^{\ell-2}\frac{r_j^k-1}{r_j^{k+i}(r_j-1)}+1\right).\end{aligned}$$

Moreover, observe that

$$\begin{aligned}\frac{1}{r_j}+\sum_{i=1}^{\ell-2}\frac{r_j^k-1}{r_j^{k+i}(r_j-1)}+1 &= \frac{1}{r_j}+\frac{r_j^k-1}{r_j-1}\sum_{i=1}^{\ell-2}\frac{1}{r_j^{k+i}}+1\\ &= \frac{1}{r_j}+r_j^{k+\ell-1}\sum_{i=1}^{\ell-2}\frac{1}{r_j^{k+i}}+1\\ &= \frac{r_j^\ell-1}{r_j(r_j-1)}.\end{aligned}$$

Here, to obtain the second line, we have used the fact that $1\neq r_j$ is a root of $x^{k+\ell-1}-\frac{x^k-1}{x-1}$.

Now, to conclude we must compute $\prod_{j=1}^{k+\ell-1} \frac{r_j^\ell - 1}{r_j(r_j-1)}$. To do so, let ω be a primitive ℓ-th root of unity. Then

$$\begin{aligned}\prod_{j=1}^{k+\ell-1} (r_j^\ell - 1) &= \prod_{j=1}^{k+\ell-1} \prod_{i=1}^{\ell} (r_j - \omega^i) = \prod_{i=1}^{\ell} \prod_{j=1}^{k+\ell-1} (r_j - \omega^i) \\ &= \left(\prod_{j=1}^{k+\ell-1} (r_j - 1)\right) \left(\prod_{i=1}^{\ell-1} \prod_{j=1}^{k+\ell-1} (r_j - \omega^i)\right) \\ &= \left(\prod_{j=1}^{k+\ell-1} (r_j - 1)\right) (-1)^{(k+\ell-1)(\ell-1)} \left(\prod_{i=1}^{\ell-1} g_{k,\ell}(\omega^i)\right).\end{aligned}$$

Furthermore, $g_{k,\ell}(\omega^i) = \omega^{i(k+\ell-1)} - \frac{\omega^{ik}-1}{\omega^i - 1} = -\frac{\omega^{i(k-1)}-1}{\omega^i-1}$. Consequently, we have

$$\prod_{j=1}^{k+\ell-1} \frac{r_j^\ell - 1}{r_j(r_j-1)} = (-1)^{(k+\ell)\ell} \left(\prod_{i=1}^{\ell-1} \frac{\omega^{i(k-1)} - 1}{\omega^i - 1}\right).$$

Finally observe that

$$\prod_{i=1}^{\ell-1} \frac{\omega^{i(k-1)} - 1}{\omega^i - 1} = \begin{cases} 1, & \text{if } \gcd(\ell, k-1) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

We can now collect all the work done to obtain the following result.

Theorem. *For all integers k and ℓ with $k, \ell \geq 2$, we have*

$$\det(\widetilde{A}_{0,k,\ell}) = \begin{cases} (-1)^{\frac{(k+\ell)(k+\ell-1)}{2}+1}, & \text{if } \ell - 1 = \alpha k \text{ and } \gcd(\ell, k-1) = 1; \\ (-1)^{\frac{(k+\ell)(k+\ell-1)}{2}}, & \text{if } \ell - 2 = \beta k \text{ and } \gcd(\ell, k-1) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary. *Let $k_0, \ell_0 \geq 2$ be any integers. Then the following hold:*

i) *The sequence $\{\alpha_\ell\}_{\ell \geq 2}$ given by $\alpha_\ell = |\det(\widetilde{A}_{0,k_0,\ell})|$ is periodic, and its period is a divisor of $k_0 \cdot \mathrm{rad}(k_0 - 1)$.*
ii) *The sequence $\{\beta_k\}_{k \geq k}$ given by $\beta_k = |\det(\widetilde{A}_{0,k,\ell_0})|$ is eventually zero.*

Proof. i) Clearly $\gcd(\ell, k_0 - 1) > 1$ implies that $\gcd(\ell + k_0 \cdot \mathrm{rad}(k_0 - 1), k_0 - 1) > 1$. In the same way, if $\ell - 1$ and $\ell - 2$ are not multiples of k_0, then neither are $\ell + k_0 \cdot \mathrm{rad}(k_0 - 1) - 1$ or $\ell + k_0 \cdot \mathrm{rad}(k_0 - 1) - 2$. Consequently, if $\alpha_\ell = 0$, also $\alpha_{\ell + k_0 \cdot \mathrm{rad}(k_0-1)} = 0$ as claimed.

ii) If $k \geq \ell_0$ obviously neither $\ell - 1$ nor $\ell - 2$ can be multiples of k and therefore $\beta_k = 0$ for every $k \geq \ell_0$.

□

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